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STRUCTURAL RELIABILITY THEORY
PAPER NO. 152

Submitted to Journal of Sound and Vibration

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ABSTRACT

Differential equations are derived which exactly govern the evolution of the second-order response moments of a single-degree-of-freedom (SDOF) bilinear hysteretic oscillator subject to stationary Gaussian white noise excitation. Then, considering cases for which response stationarity will be achieved, i.e., excluding the case of an elastic-perfectly-plastic oscillator, algebraic equations for the response moments are found. By the nature of the problem, these moments depend on the probability of the oscillator being in the plastic state. Upon considering oscillators with low yield levels and using analytically-available information, physical reasoning, and approximations supported by empirical observation, an equation for the probability of the oscillator being in the plastic state is derived. Upon numerical solution of this equation, analytical approximations to the response moments can be obtained. All analytical, approximate, and numerical results are verified by extensive Monte Carlo simulations.

1. INTRODUCTION

Over the last few decades, the problems of predicting the response and reliability of hysteretic systems subject to random excitation have received considerable attention. While this attention has mainly stemmed from the engineering usefulness of hysteretic systems in the modelling of actual physical and mechanical phenomena, it has also been due to the inherent difficulty in obtaining exact closed-form solutions to these problems. As a consequence, various approximate analytical procedures for handling the nonlinearity and non-analyticity of hysteretic systems have been proposed. Among the approximate analytical methods most commonly used include Markov methods[1,2,3,4,5,6,7], equivalent linearization[8,9,10,11,12,13,14,15,16,17], equivalent nonlinearization with cumulant-neglect closure[18], the associated linear oscillator approach[19,20,21,22,23,24,25,26], and the Slepian process approach[27,28,29]. (It should be noted that due to the enormous amounts of literature on hysteretic systems, the foregoing is by no means an exhaustive list of references on the subject.) While most of the above-cited techniques produce good-to-excellent results for weakly-to-moderately nonlinear oscillators, when highly-nonlinear oscillators are considered, such as when the yield level is low relative to the standard deviation of the corresponding linear oscillator and, in addition, the secondary-to-primary stiffness is small, the accuracy of most of the aforementioned techniques breaks down or the methods are not justifiably applicable. In such cases, Monte Carlo methods are often the only recourse to solving the problem.

In this paper, we consider a bilinear hysteretic oscillator subject to stationary Gaussian white noise excitation. Such an oscillator is often used as an idealized model of a simple structure undergoing earthquake excitations. As such, it may often be necessary to determine approximate, yet accurate, statistics of the oscillator's stationary response–especially the standard deviation of displacement–quickly and efficiently. To this end, a system of differential equations is derived which exactly governs the second-order response moments of the bilinear hysteretic oscillator. Then, considering cases of response stationarity, that is, excluding the case of an elasto-plastic oscillator, the differential equations can be reduced to algebraic equations which depend not only on the unconditional moments of response but also on the probability of the oscillator being in the plastic state as well as on response moments conditioned on the oscillator being in the plastic state. Using physical reasoning supported by empirical observation for the case of low yield levels, *a priori* bounds on the conditional response moments are obtained. These bounds in conjunction with analytical and empirical approximations of unprovided response moments leads to an equation for the probability of being in the plastic state. Upon numerical solution of this equation, analytical approximations of the unconditional standard deviations of velocity and displacement follow immediately. The accuracy of these approximations as well as the accuracy of all analytical, approximate, and numerical results contained herein is verified through extensive Monte Carlo simulations.

2. MOMENT EQUATIONS

Consider the following equations of motion for a SDOF bilinear hysteretic oscillator excited by Gaussian white noise[30],

$$\ddot{X} + 2\xi\omega_0\dot{X} + \omega_0^2(\alpha X + (1-\alpha)Z) = W(t) \tag{1}$$

$$\dot{Z} = \dot{X}\{1 - H(\dot{X})H(Z-u) - H(-\dot{X})H(-Z-u)\} \tag{2}$$

$$X(0) = \dot{X}(0) = Z(0) = 0 \tag{3}$$

where $X, \dot{X}$, and $\ddot{X}$ denote, respectively, the random displacement, velocity, and acceleration of the oscillator; Z, the random hysteretic component of the response; $W(t)$, a zero-mean Gaussian white noise excitation with double-sided spectral density S_0; ξ, the damping ratio; ω_0, the circular eigenfrequency of the corresponding linear oscillator; α, the secondary-to-primary stiffness ratio; u, the positive yield level of the oscillator; and $H(\cdot)$, a Heaviside step-function defined as

$$H(y) = \begin{cases} 1, & \text{if } \; y \geq 0 \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

In state-vector form, the foregoing equations can be written as

$$\mathbf{Y} = \mathbf{a}(\mathbf{Y})\ dt + \mathbf{b}\ dB(t) \tag{5}$$

where

$$\mathbf{Y} = \begin{bmatrix} X \\ \dot{X} \\ Z \end{bmatrix}, \quad \mathbf{a}(\mathbf{Y}) = \begin{bmatrix} \dot{X} \\ -2\xi\omega_0\dot{X} - \omega_0^2(\alpha X + (1-\alpha)Z) \\ \dot{X}\{1 - H(\dot{X})H(Z-u) - H(-\dot{X})H(-Z-u)\} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{6}$$

and $B(t)$ is a Wiener process. The vector $\mathbf{a}$ is referred to as the drift vector and the vector $\mathbf{b}$, in general, is known as the diffusion vector.

Equation (5) represents a vector Markov diffusion process interpreted in the Itô sense and as such, Itô's differential formula can be applied to derive equations which govern the evolution of the response moments. Itô's differential formula for an arbitrary, well-behaved function $f(\mathbf{Y}, t)$ of the state vector $\mathbf{Y}$ and time t is given by

$$df = \frac{\partial f}{\partial t}dt + \sum_{i=1}^{n} \frac{\partial f}{\partial Y_i}dY_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f}{\partial Y_i \partial Y_j}dY_i dY_j \tag{7}$$

By substituting (5) into (7), letting $f = Y_{i_1}^{m_1} Y_{i_2}^{m_2} \cdots Y_{i_k}^{m_k}$, and taking the expectation, differential equations for the mth-order joint moments of response quantities can be derived, where $m = \sum_j m_j$. The resulting general expressions for first- and second-order joint moments are given by

$$\dot{\mu}_i = \mathrm{E}[a_i] \tag{8}$$

$$\dot{\kappa}_{ij} = \mathrm{E}[a_i Y_j] + \mathrm{E}[a_j Y_i] + b_i b_j \tag{9}$$

where E is the expectation operator, $\mu_i = \mathrm{E}[Y_i]$ and $\kappa_{ij} = \mathrm{E}[Y_i Y_j]$, and Y_j, a_j, and b_j denote the jth components of the vectors $\mathbf{Y}$, $\mathbf{a}$, and $\mathbf{b}$, respectively.

Due to the asymmetry of the drift vector, namely,

$$\mathbf{a}(\mathbf{Y}) = -\mathbf{a}(-\mathbf{Y}) \tag{10}$$

$\mathrm{E}[a_i] = 0$ for each i as a result of the zero initial conditions. Then, Equation (8) implies that the vector of first-order response quantities is identically zero, i.e., $\mu_i = 0$ for each i. Thus, for reasons of practicability, only second-order response moments will be considered.

Substituting (6) into (9) yields the following set of differential equations governing the time-evolution of the second-order joint moments of response (dependence of the joint moments on time is suppressed for notational convenience):

$$\begin{aligned}
\dot{\kappa}_{11} &= 2\kappa_{12} \\
\dot{\kappa}_{12} &= \kappa_{22} - 2\xi\omega_0\kappa_{12} - \omega_0^2(\alpha\kappa_{11} + (1-\alpha)\kappa_{13}) \\
\dot{\kappa}_{13} &= \kappa_{23} + \kappa_{12} - \mathrm{E}[X\dot{X}H(\dot{X})H(Z-u)] - \mathrm{E}[X\dot{X}H(-\dot{X})H(-Z-u)] \\
\dot{\kappa}_{22} &= -4\xi\omega_0\kappa_{22} - 2\omega_0^2(\alpha\kappa_{12} + (1-\alpha)\kappa_{23}) + 2\pi S_0 \\
\dot{\kappa}_{23} &= -2\xi\omega_0\kappa_{23} - \omega_0^2(\alpha\kappa_{13} + (1-\alpha)\kappa_{33}) + \\
&\qquad \kappa_{22} - \mathrm{E}[\dot{X}^2 H(\dot{X})H(Z-u)] - \mathrm{E}[\dot{X}^2 H(-\dot{X})H(-Z-u)] \\
\dot{\kappa}_{33} &= 2\kappa_{23} - 2\mathrm{E}[\dot{X}ZH(\dot{X})H(Z-u)] - 2\mathrm{E}[\dot{X}ZH(-\dot{X})H(-Z-u)]
\end{aligned} \tag{11}$$

where $\kappa_{11} = \mathrm{E}[X^2], \kappa_{12} = \mathrm{E}[X\dot{X}], \kappa_{13} = \mathrm{E}[XZ], \kappa_{22} = \mathrm{E}[\dot{X}^2], \kappa_{23} = \mathrm{E}[\dot{X}Z]$, and $\kappa_{33} = \mathrm{E}[Z^2]$.

In most approximate analyses involving moment equations, the expectations above involving Heaviside functions are replaced by equivalent polynomial expansions of various orders. Here these expectations will be expressed as response moments conditioned on the oscillator being in the plastic state. To this end, let $p_{X\dot{X}Z}(x, \dot{x}, z)$ represent the joint probability density function of $X, \dot{X}$, and Z. This joint probability density is of the mixed type in the sense that X and $\dot{X}$ are continuous variables of infinite range whereas Z ranges only between $-u$ and u and has a finite probability of assuming the values $-u$ or u. In addition, let $P\{Z = u\}$ represent the probability that the oscillator has reached the positive yield state. Due to the symmetry of the oscillator, the probability of being in the positive yield state is equal to the probability of being in the negative yield state, i.e., $P\{Z = u\} = P\{Z = -u\}$. With these conventions established, consider the term $\mathrm{E}[X\dot{X}H(\dot{X})H(Z-u)]$. This term can be rewritten as follows

$$\begin{aligned}
&\mathrm{E}[X\dot{X}H(\dot{X})H(Z-u)] = \\
&\int_{-\infty}^{\infty} dx\; x \int_{-\infty}^{\infty} d\dot{x}\; \dot{x}\; H(\dot{x}) \int_{-u^-}^{u^+} dz\; H(z-u)\, p_{X\dot{X}Z}(x, \dot{x}, z) = \\
&\int_{-\infty}^{\infty} dx\; x \int_{0}^{\infty} d\dot{x}\; \dot{x}\; p_{X\dot{X}|Z}(x, \dot{x}|z=u)\; P\{Z=u\} = P\{Z=u\}\mathrm{E}[X\dot{X}|Z=u]
\end{aligned} \tag{12}$$

Intuitively, the result is clear, the term $X\dot{X}H(\dot{X})H(Z-u)$ is equal to $X\dot{X}$ if the system is in the positive yield state and zero otherwise. Note that the velocity of the oscillator is non-negative when $Z = u$. Due to the symmetry of the oscillator, expressions similar to (12) are obtained for the following expectations:

$$\begin{aligned}
&\mathrm{E}[X\dot{X}H(-\dot{X})H(-Z-u)] = P\{Z=-u\}\mathrm{E}[X\dot{X}|Z=-u] = P\{Z=u\}\mathrm{E}[X\dot{X}|Z=u] \\
&\mathrm{E}[\dot{X}^2H(\dot{X})H(Z-u)] = \mathrm{E}[\dot{X}^2H(-\dot{X})H(-Z-u)] = P\{Z=u\}\mathrm{E}[\dot{X}^2|Z=u] \\
&\mathrm{E}[\dot{X}ZH(\dot{X})H(Z-u)] = \mathrm{E}[\dot{X}ZH(-\dot{X})H(-Z-u)] = uP\{Z=u\}\mathrm{E}[\dot{X}|Z=u]
\end{aligned} \tag{13}$$

Substituting the results of (12) and (13) into (11) and setting the left-hand sides equal to zero, i.e., considering the state of stationarity, yields the following algebraic equations for the determination of the stationary response moments

$$\begin{aligned}
&0 = 2\kappa_{12} \\
&0 = \kappa_{22} - 2\xi\omega_0\kappa_{12} - \omega_0^2(\alpha\kappa_{11} + (1-\alpha)\kappa_{13}) \\
&0 = \kappa_{23} + \kappa_{12} - 2P\{Z=u\}\mathrm{E}[X\dot{X}|Z=u] \\
&0 = -4\xi\omega_0\kappa_{22} - 2\omega_0^2(\alpha\kappa_{12} + (1-\alpha)\kappa_{23}) + 2\pi S_0 \\
&0 = -2\xi\omega_0\kappa_{23} - \omega_0^2(\alpha\kappa_{13} + (1-\alpha)\kappa_{33}) + \kappa_{22} - 2P\{Z=u\}\mathrm{E}[\dot{X}^2|Z=u] \\
&0 = 2\kappa_{23} - 4uP\{Z=u\}\mathrm{E}[\dot{X}|Z=u]
\end{aligned} \tag{14}$$

The first of equations (14) eliminates κ_{12} from consideration. Further, both the third and last of equations (14) yield expressions for κ_{23}. Use will be made of the latter, as it requires only the knowledge of the marginal probability density of $\dot{X}$ when the system is in the plastic state, whereas the former requires information about the joint probability density of X and $\dot{X}$ in the plastic state. Note that by discarding the third of equations (14), four nontrivial/non-redundant equations for the eight unknowns κ_{11}, κ_{13}, κ_{22}, κ_{23}, κ_{33}, $\mathrm{E}[\dot{X}|Z=u]$, $\mathrm{E}[\dot{X}^2|Z=u]$, and $P\{Z=u\}$ remain. Hence extra information regarding the physical system, including the probability density of $\dot{X}$ during plastic excursions, is needed to obtain numerical results for the response moments.

3. MARGINAL PROBABILITY DENSITY FUNCTION OF VELOCITY DURING PLASTIC EXCURSIONS

Consider the stationary response of the corresponding linear SDOF dynamical system. Let u be an arbitrary positive distance from the origin . It is well-known from crossing-theory, that if values of $\dot{X}$ are sampled at u-upcrossings, the empirical distribution of $\dot{X}$ tends to a Rayleigh distribution. That is, the probability density of $\dot{X}$ tends to one which has no probability mass arbitrarily near zero as the velocity of the system must be greater than zero in order to have a u-upcrossing.

From the equations of motion of the bilinear hysteretic oscillator (1), it is seen that between yield-level excursions, the system acts as a linear oscillator. Thus, if the yield level is sufficiently high in relation to the standard deviation of the corresponding linear oscillator, it is reasonable to assume that between plastic excursions the oscillator's response renormalizes to Gaussianity such that the distribution of $\dot{X}$ at yield-level upcrossings is approximately Rayleigh distributed with mean $\sqrt{\pi/2}\sigma_{\dot{X}}$ and second moment $2\sigma^2_{\dot{X}}$, where $\sigma_{\dot{X}}$ is the unconditional standard deviation of velocity (this idea forms the basis of the Slepian process approach). However, this distribution does not account for the distribution of $\dot{X}$ for the system's entire sojourn to the plastic state. During plastic excursions, if white noise effects are neglected, the velocity of the oscillator monotonically decreases from its value at yield-level-upcrossing to zero at which point the oscillator exits the plastic state. Physically, this means that the average value of $\dot{X}$ during plastic excursions is less than the average value of $\dot{X}$ at yield-level upcrossings. Further, as every yield-level upcrossing corresponds to a zero-velocity yield-level downcrossing, it seems physically reasonable to expect the probability density of $\dot{X}$ during plastic excursions to have some probability mass arbitrarily near zero – possibly approaching a half-Gaussian probability density with mean $\sqrt{2/\pi}\sigma_{\dot{X}}$ and second moment $\sigma^2_{\dot{X}}$.
For low yield levels (in relation to the standard deviation of the corresponding linear oscillator), it is not the case that the system's response will renormalize to Gaussianity between plastic excursions. However, the velocity at a u-upcrossing still must be positive and this velocity will again steadily decrease to zero during plastic excursions such that the average velocity during plastic excursions is less than the average value at u-upcrossings. Thus, while the distribution of the velocity at u-upcrossings is not expected to be Rayleigh, it is conjectured that the mean and second moment of the Rayleigh distribution (with parameter $\sigma^2_{\dot{X}}$ equal to the unconditional variance of velocity of the bilinear oscillator) form upper bounds on the mean and second moment of $\dot{X}$ during plastic excursions. Further, it is conjectured (again for low yield levels) that the mean and second moment of the half-Gaussian density (with parameter $\sigma^2_{\dot{X}}$ equal to the unconditional variance of velocity) form lower bounds on the mean and second moment of $\dot{X}$ during plastic excursions.

These conjectures are supported by results obtained through Monte Carlo simulation. A SDOF bilinear hysteretic with $\omega_0 = 1$, $\xi = 0.05$, and $\alpha = 1/21$ was considered for various yield levels u. The intensity of the white noise excitation was prescribed so that the mean-square displacement of the corresponding linear oscillator, $\sigma^2_{X,0}$, was 1, i.e., $S_0 = 2\xi\omega_0^3/\pi$. Thus, the yield level u is normalized in proportion to the standard deviation of the corresponding linear oscillator.

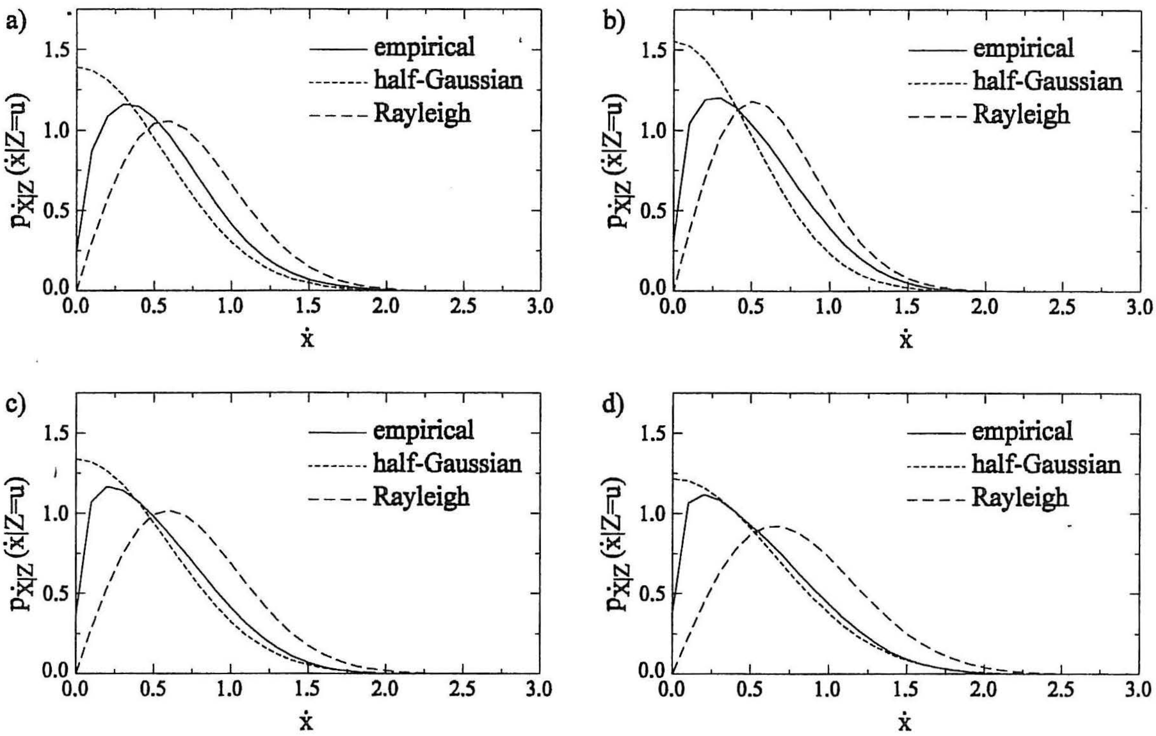

Figure 1: Conditional probability densities of $\dot{X}$ given $Z = u$ found by Monte Carlo simulations in comparison with half-Gaussian and Rayleigh densities; (a) $u/\sigma_{X,0} = 0.25$; (b) $u/\sigma_{X,0} = 0.50$; (c) $u/\sigma_{X,0} = 0.75$; (d) $u/\sigma_{X,0} = 1.00$.

Figure 1 shows the conditional probability density functions of $\dot{X}$ given that the system is in the positive yield state in comparison with Rayleigh and half-Gaussian densities for various values of $u/\sigma_{X,0}$. The means and second moments of the distributions are listed in Table 1.

For ratios of $u/\sigma_{X,0}$ less than or equal to one, the means and second moments of the Rayleigh and half-Gaussian distributions form upper and lower bounds, respectively, on the means and second moments of the empirical distributions of $\dot{X}$. Thus in the ensuing analysis, the terms $\mathrm{E}[\dot{X}|Z = u]$ and $\mathrm{E}[\dot{X}^2|Z = u]$ in equations (14) are taken to be bounded as follows:

$$\sqrt{\frac{2}{\pi}}\sigma_{\dot{X}} \quad \leq \quad \mathrm{E}[\dot{X}|Z = u] \quad \leq \quad \sqrt{\frac{\pi}{2}}\sigma_{\dot{X}} \tag{15}$$

$$\sigma^2_{\dot{X}} \quad \leq \quad \mathrm{E}[\dot{X}^2|Z = u] \quad \leq \quad 2\sigma^2_{\dot{X}} \tag{16}$$

Table 1: Means and second moments of the distributions appearing in Figure 1

$u/\sigma_{X,0}$	Means			Second Moments		
	Half-Gaussian	Empirical	Rayleigh	Half-Gaussian	Empirical	Rayleigh
0.250	0.457	0.553	0.718	0.328	0.436	0.656
0.500	0.409	0.517	0.643	0.263	0.385	0.526
0.750	0.475	0.531	0.746	0.354	0.415	0.708
1.000	0.523	0.554	0.821	0.429	0.458	0.858

4. EQUATION FOR THE PROBABILITY OF BEING IN THE PLASTIC STATE

For ease of notation consider the following reformulation of equation (1):

$$\ddot{X} + a\dot{X} + bX + cZ = W(t) \tag{17}$$

where $a = 2\xi\omega_0$, $b = \omega_0^2\alpha$, and $c = \omega_0^2(1-\alpha)$. The nontrivial/nonredundant algebraic equations for the stationary response moments become

$$\begin{aligned}
0 &= \kappa_{22} - b\kappa_{11} - c\kappa_{13} \\
0 &= \kappa_{22} + \frac{c}{a}\kappa_{23} - \frac{\pi S_0}{a} \\
0 &= -a\kappa_{23} - b\kappa_{13} - c\kappa_{33} + \kappa_{22} - 2P\{Z=u\}\mathrm{E}[\dot{X}^2|Z=u] \\
0 &= 2\kappa_{23} - 4uP\{Z=u\}\mathrm{E}[\dot{X}|Z=u]
\end{aligned} \tag{18}$$

The last of equations (18) indicates that $\kappa_{23} = 2uP\{Z=u\}\mathrm{E}[\dot{X}|Z=u]$ which, given the bounds on $\mathrm{E}[\dot{X}|Z=u]$, can be written as

$$\kappa_{23} = 2\;u\;P\{Z=u\}\;d\;\sigma_{\dot{X}} = f\;\sigma_{\dot{X}} \qquad \sqrt{2/\pi} \leq d \leq \sqrt{\pi/2} \tag{19}$$

where $f \equiv 2\,u\,P\{Z=u\}\,d$. Substituting this result into the second of equations (18) and using $\kappa_{22} = \mathrm{E}[\dot{X}^2] = \sigma_{\dot{X}}^2$ yields a quadratic equation in $\sigma_{\dot{X}}$, namely

$$\sigma_{\dot{X}}^2 + \frac{cf}{a}\sigma_{\dot{X}} - \frac{\pi S_0}{a} = 0 \tag{20}$$

whose positive root is given by

$$\sigma_{\dot{X}} = -\frac{cf}{2a} + \sqrt{\left(\frac{cf}{2a}\right)^2 + \frac{\pi S_0}{a}} \tag{21}$$

The standard deviation of velocity as given by equation (21) is a monotonically decreasing function of $udP\{Z = u\}$ for given system parameters. It follows then that the mean-square velocity, $\sigma_{\dot{X}}^2$ will also be a monotonically decreasing function of $udP\{Z = u\}$. When $u \to 0$, $P\{Z = u\} \to 0.5$ and the system behaves as a linear oscillator with stiffness $\alpha\omega_0^2$. When $u \to \infty$, $P\{Z = u\} \to 0$ and the system behaves as a linear oscillator with stiffness ω_0^2. In both cases, $\sigma_{\dot{X}}^2 = \pi S_0/2\xi\omega_0$, which forms an upper bound on the mean-square velocity. When $udP\{Z = u\}$ is nonzero, the oscillator's stiffness will be repeatedly softened upon its entries into the plastic state. This softening of the restoring force leads to oscillations of lower frequency and, consequently, lower velocity implying that the mean-square velocity will decrease.

Turning now to the third of equations (18) and using $\kappa_{22} = \sigma_{\dot{X}}^2$, $\kappa_{23} = f\sigma_{\dot{X}}$, and equation (16) results in

$$\sigma_{\dot{X}}^2(1 - g) - af\sigma_{\dot{X}} = b\kappa_{13} + c\kappa_{33} \tag{22}$$

where

$$g \equiv 2P\{Z = u\}e \tag{23}$$

$$1 \leq e \leq 2 \tag{24}$$

The significance of equation (22) is that the left-hand side is entirely a function of known system constants, parameters that are bounded, and the probability of being in the plastic state. Consequently, if the right-hand side of equation (22) *a priori* can be well-approximated in terms of system constants and/or the probability of being in the plastic state, the equation can be solved for $P\{Z = u\}$ with regard to the bounded constants d and e to give functioning (though not rigorous) bounds on $P\{Z = u\}$. Thus, the problem becomes one of determining valid approximations of κ_{13} and κ_{33}.

5. APPROXIMATIONS TO κ_{33} AND κ_{13}

First consider κ_{33}. Let $p_Z(z)$ be the marginal mixed probability density of z and let $\hat{p}_Z(z)$ represent the continuous part of $p_Z(z)$ between $-u$ and u. Due to the symmetry of the oscillator, $p_Z(z)$ is symmetric about zero. Assuming $\hat{p}_Z(z)$ to be approximately uniformly distributed, i.e., $\hat{p}_Z(z) \approx \frac{1}{2u}(1 - 2P\{Z = u\})$, it follows that

$$\begin{aligned}
&\kappa_{33} = \mathrm{E}[Z^2] = 2\int_0^{u^+} dz\ z^2\ p_Z(z) = \\
&u^2 2P\{Z = u\} + 2\int_0^{u} dz\ z^2\ \hat{p}_Z(z) \approx \\
&u^2 2P\{Z = u\} + u^2\frac{1}{3}(1 - 2P\{Z = u\})
\end{aligned} \tag{25}$$

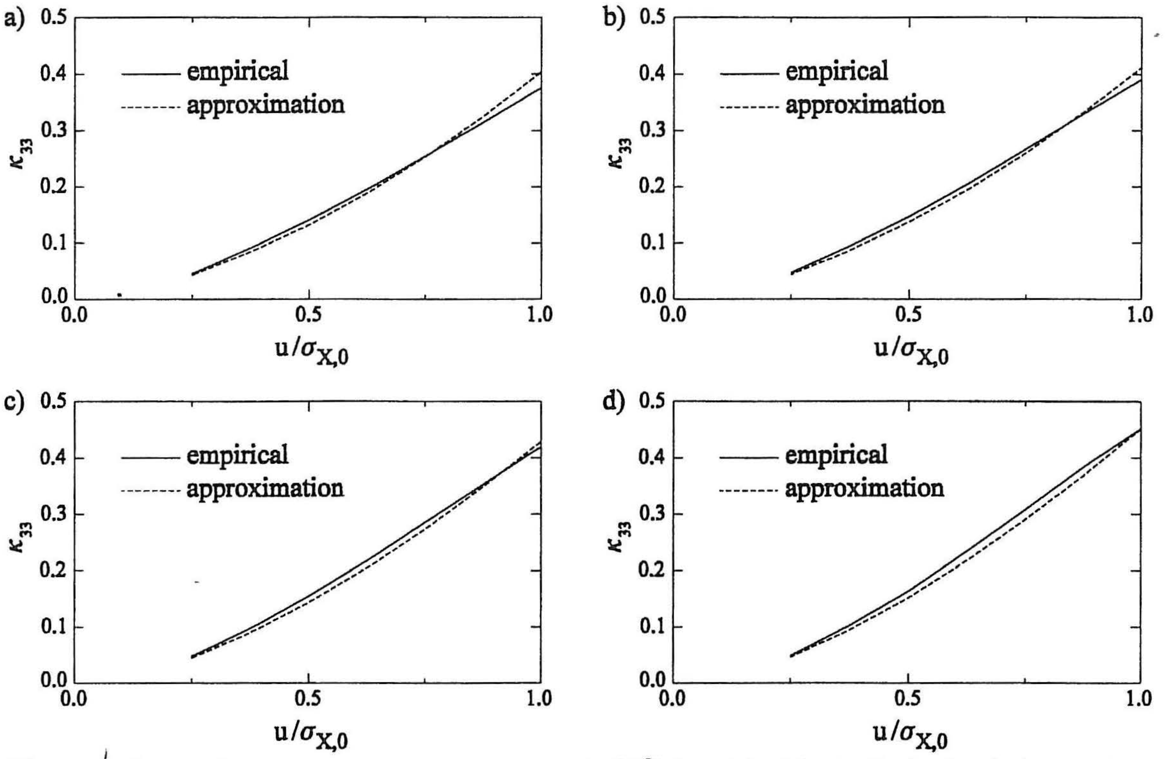

Figure 2: Comparison of estimated exact values of $\mathrm{E}[Z^2]$ found by Monte Carlo simulations with approximate values found using eq. (25); (a) $\alpha = 0.05$; (b) $\alpha = 0.25$; (c) $\alpha = 0.50$; (d) $\alpha = 0.75$.

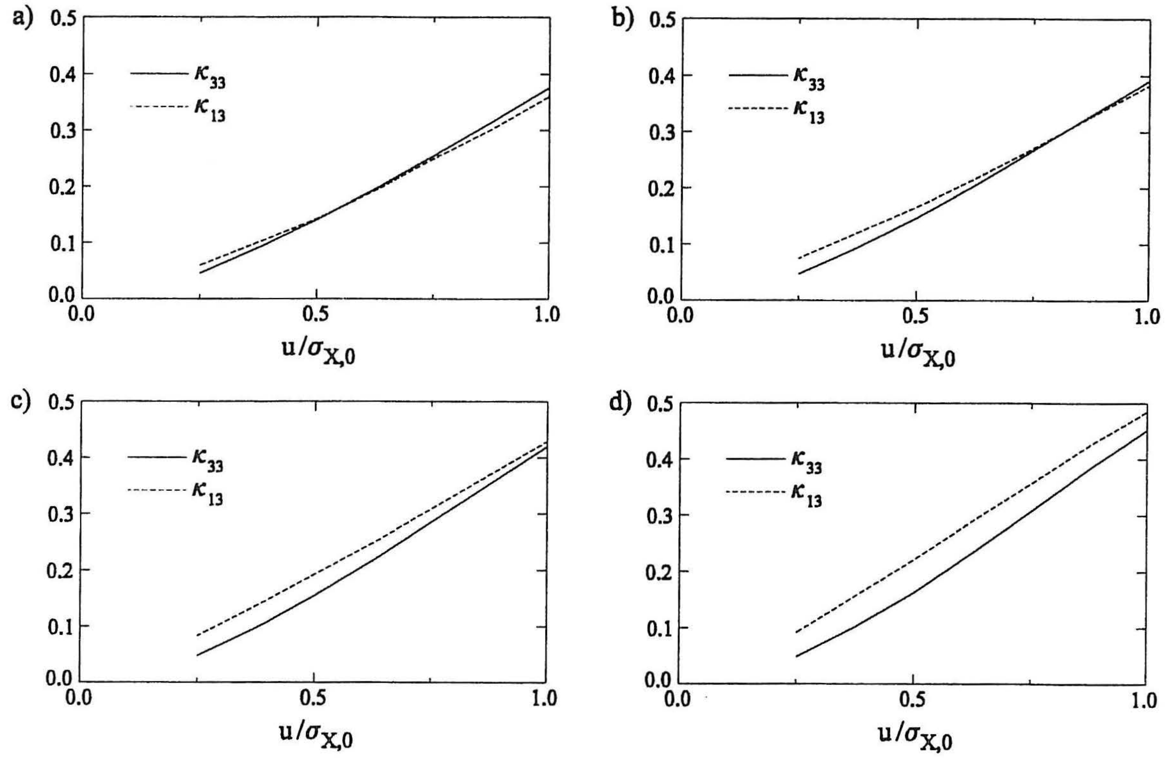

Figure 3: Comparison of estimated exact values of $\mathrm{E}[Z^2]$ with estimated exact values of $\mathrm{E}[XZ]$ obtained by Monte Carlo simulations; (a) $\alpha = 0.05$; (b) $\alpha = 0.25$; (c) $\alpha = 0.50$; (d) $\alpha = 0.75$.

Figure 2 shows estimated exact values of κ_{33} as computed via Monte Carlo simulations in comparison with values predicted by using the estimated exact values of $P\{Z = u\}$ as determined by Monte Carlo simulations. As is evident, (25) provides a very close approximation to the actual value of κ_{33}, especially at low values of $u/\sigma_{X,0}$.

To determine an approximation for κ_{13}, first note that $X = Z + \Delta$ where Δ is the net plastic displacement of the oscillator. Thus,

$$\kappa_{13} = \mathrm{E}[XZ] = \mathrm{E}[Z^2] + \mathrm{E}[Z\Delta] \approx \mathrm{E}[Z^2] = \kappa_{33} \tag{26}$$

where use was made of the fact, uncovered by extensive Monte Carlo simulations, that Z and Δ are approximately uncorrelated. This correlation is the difference between κ_{13} and κ_{33} but it is quite small for values of $\alpha \leq 0.25$ (which covers most cases of practical interest) and reasonably small for $\alpha \geq 0.50$ as can be seen in Figure 3.

Thus, in the sequel the right-hand side of equation (22) is approximated by

$$b\kappa_{13} + c\kappa_{33} \approx b\kappa_{33} + c\kappa_{33} \approx (b+c)u^2 \left(2P\{Z = u\} + \frac{1}{3}(1 - 2P\{Z = u\})\right) \tag{27}$$

With the above results in conjunction with equation (22), the following equation for the probability of being in the plastic state is obtained,

$$\sigma_{\dot{X}}^2(1-g) - af\sigma_{\dot{X}} - (b+c)u^2 \left(2P\{Z = u\} + \frac{1}{3}(1 - 2P\{Z = u\})\right) = 0 \tag{28}$$

where $\sigma_{\dot{X}}$ is given in terms of $P\{Z = u\}$ by equation (21).

Equation (28) can be reduced to a quartic equation in $P\{Z = u\}$ and thus has an exact analytical solution. However, it is extremely lengthy as verified by the software program *Mathematica*. Thus, for purposes of numerical evaluation, equation (28) was solved numerically using *Mathematica*'s root-finding algorithm.

6. NUMERICAL EXAMPLES

To investigate the accuracy of the results given by equation (28) its dependence on the yield level, u, and the secondary-to-primary stiffness ratio, α, various parametric cases were considered. For each case, the intensity of the white noise, S_0, was prescribed so that the mean-square displacement of the corresponding linear oscillator was equal to one. Then u could be interpreted as the ratio of the yield level to the standard deviation of the corresponding linear oscillator or, equivalently, the reciprocal of u could be viewed as the factor by which the white noise intensity was increased or decreased with respect to the linear case. Values of u considered ranged from 0.25 to 1.0 while values of α ranged from 0.05 to 0.75. In addition, recall that the parameters d and e enter (28) through f and g and these parameters are restricted to the intervals $\sqrt{\frac{2}{\pi}} \leq d \leq \sqrt{\frac{\pi}{2}}$, $1 \leq e \leq 2$.

$P\{Z = u\}$ was evaluated in two different ways with respect to d and e, namely, by setting d and e to their extreme low values and extreme high values, respectively. For all cases considered, $\omega_0 = 1.0\ s^{-1}$ (without loss of generality as time can always be measured in units of $1/\omega_0$) and $\xi = 0.05$.

For all cases of α considered, the values of $P\{Z = u\}$ found by setting d and e equal to their extreme low values ($\sqrt{2/\pi}$ and 1, respectively) were on the high side, that is, they formed upper bounds on the probability of being in the plastic state. These upper bounds became increasingly sharp as $u/\sigma_{X,0}$ approached 1. The opposite was true for the values of $P\{Z = u\}$ found by setting d and e equal to their extreme high values ($\sqrt{\pi/2}$ and 2, respectively). In this case, the results formed lower bounds for all values of $u/\sigma_{X,0}$. Similarly, these bounds became increasingly sharp as $u/\sigma_{X,0}$ approached 1.

Figure 4 shows comparisons of the estimated exact values of $P\{Z = u\}$ with results found by solving equation (28) for each ordered pair of the constants d and e, namely ($\sqrt{2/\pi}$, 1) and ($\sqrt{\pi/2}$, 2).

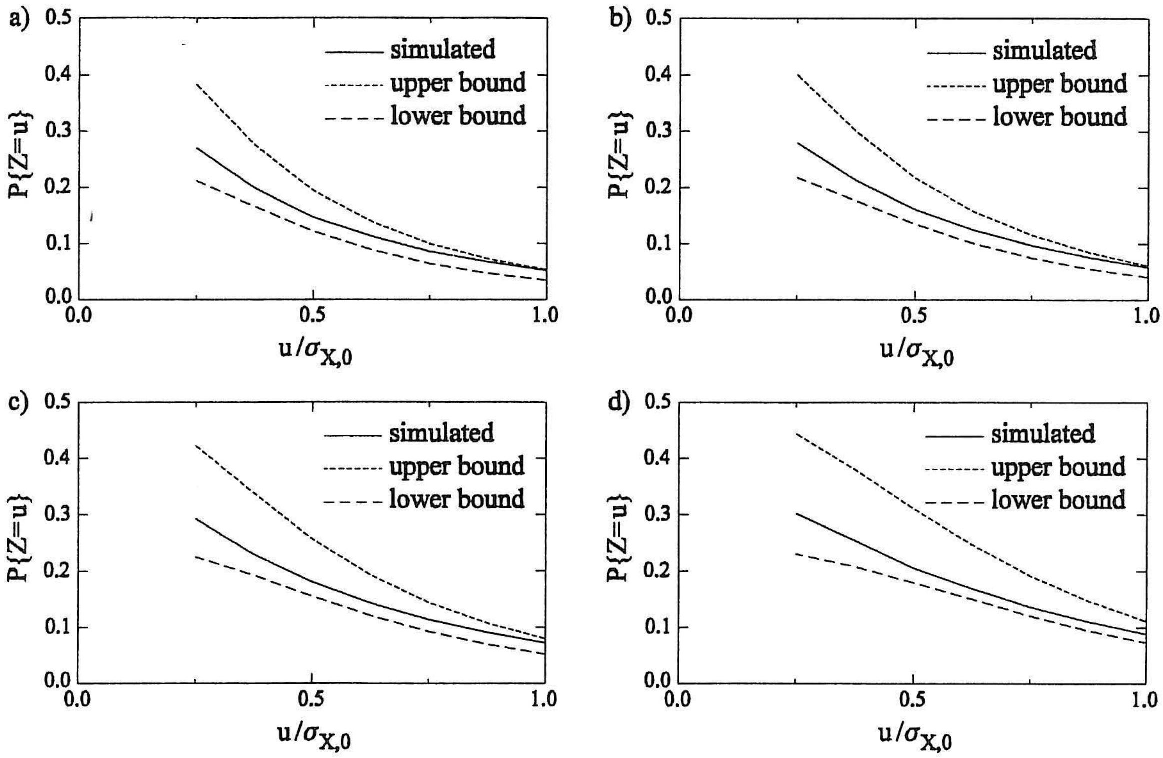

Figure 4: Solutions of (28) for the probability of being in the plastic state; (a) $\alpha = 0.05$; (b) $\alpha = 0.25$; (c) $\alpha = 0.50$; (d) $\alpha = 0.75$.

7. APPROXIMATION OF THE RESPONSE MOMENTS

With bounds on the probability of being in the plastic state established, the expression given by equation (21) can be used to determine an approximation of $\sigma_{\dot{X}}$. As before with the equation for $P\{Z = u\}$, the expression for $\sigma_{\dot{X}}$ can be evaluated using the extreme values of d and the corresponding values of $P\{Z = u\}$. These results are presented in Figure 5.

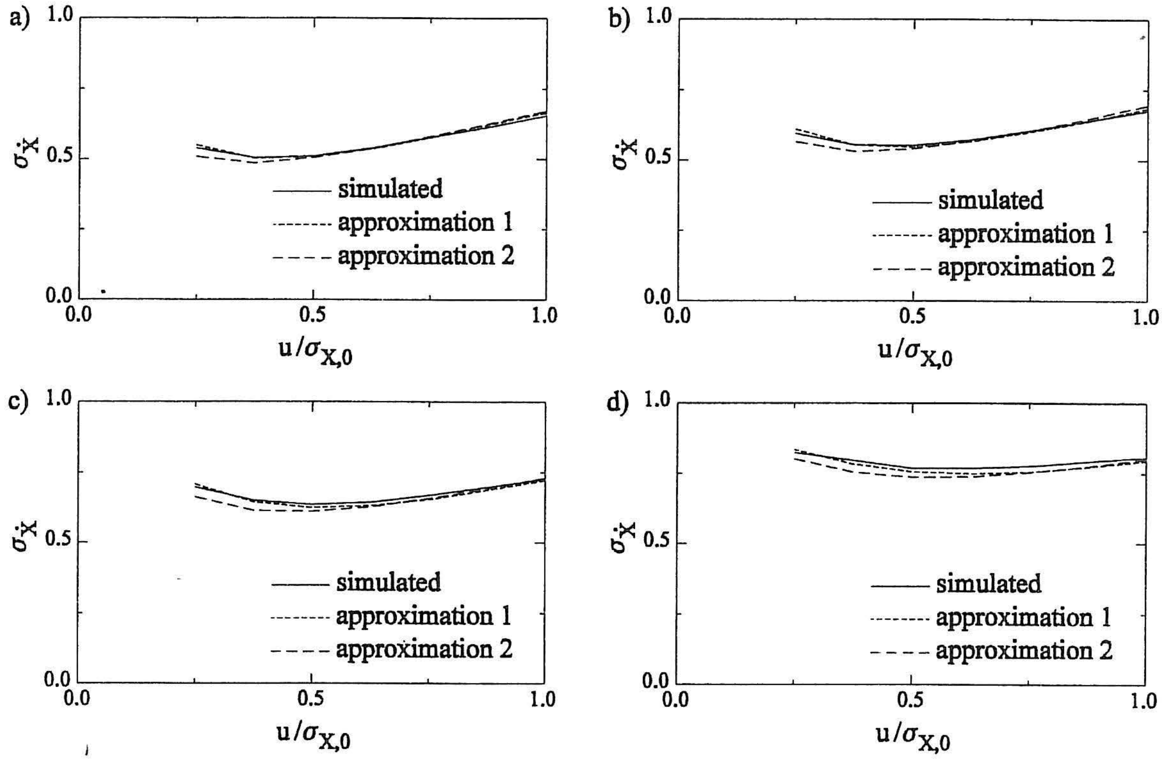

Figure 5: Comparisons of estimated exact values of $\sigma_{\dot X}$ with approximations found using eq. (21); (a) $\alpha = 0.05$; (b) $\alpha = 0.25$; (c) $\alpha = 0.50$; (d) $\alpha = 0.75$.

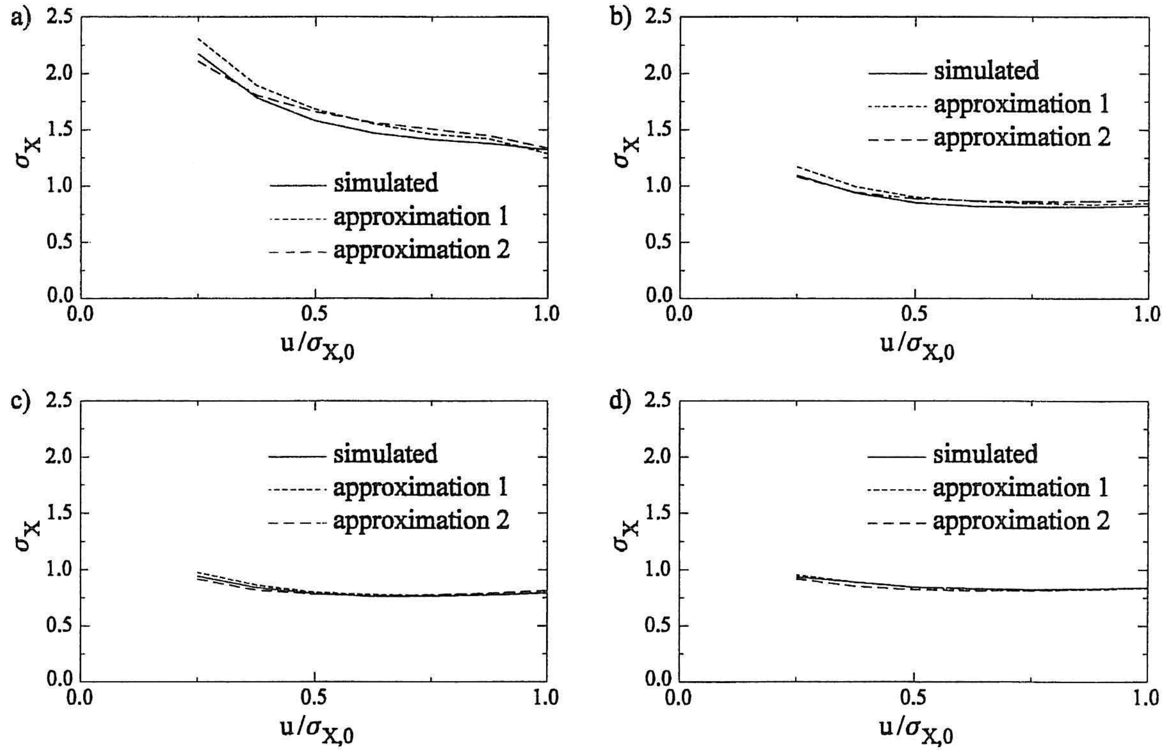

Figure 6: Comparisons of estimated exact values of σ_X with approximations found using eq. (30); (a) $\alpha = 0.05$; (b) $\alpha = 0.25$; (c) $\alpha = 0.50$; (d) $\alpha = 0.75$.

The approximate results were essentially insensitive to the values of d used as long as the corresponding value of $P\{Z = u\}$ was used as well. This is because the variables d and $P\{Z = u\}$ appear as a product in equation (21) and recall that high values of d resulted in low values of $P\{Z = u\}$ and vice-versa. Hence, the products were approximately equal. In Figure 5, "Approximation 1" corresponds to using $d = \sqrt{\pi/2}$ (and the subsequent value of $P\{Z = u\}$) whereas "Approximation 2" corresponds to using $d = \sqrt{2/\pi}$.

Similarly, an approximation of σ_X can be computed using the first of equations (18) which can be written as

$$\sigma_X^2 = \frac{1}{b}(\sigma_{\dot{X}}^2 - c\kappa_{13}) \tag{29}$$

Making use of the approximation $\kappa_{13} \approx \kappa_{33} \approx u^2\left(2P\{Z = u\} + \frac{1}{3}(1 - 2P\{Z = u\})\right)$ yields

$$\sigma_X^2 = \frac{1}{b}\sigma_{\dot{X}}^2 - \frac{c}{b}u^2\left(2P\{Z = u\} + \frac{1}{3}(1 - 2P\{Z = u\})\right) \tag{30}$$

Once again, in light of the upper and lower bounds on $P\{Z = u\}$ there are multiple ways in which to evaluate equation (30). However, a more conservative estimate is obtained by using the lower bound values of $P\{Z = u\}$ when evaluating the approximation to κ_{13} since then the right-hand side is (relatively) maximized. Figure 6 shows comparisons of the estimated actual values of σ_X with approximations found using equation (39). Here "Approximation 1" corresponds to using the value of $\sigma_{\dot{X}}$ found in "Approximation 1" above along with the minimum approximation to κ_{13}. Similarly, "Approximation 2" corresponds to using the value of $\sigma_{\dot{X}}$ found in "Approximation 2" above along with the minimum approximation to κ_{13}. As with the approximations to $\sigma_{\dot{X}}$, the approximations to σ_X very closely agree with the estimated exact values.

8. CONCLUSIONS

Algebraic equations were derived for a bilinear hysteretic oscillator excited by stationary Gaussian white noise which related stationary response moments, conditional response moments, and the probability of being in the plastic state. Then, considering the physics of the oscillator at low yield level upcrossings, *a priori* bounds, verified by Monte Carlo simulations, were assigned to the conditional response moments $\mathrm{E}[\dot{X}|Z = u]$ and $\mathrm{E}[\dot{X}^2|Z = u]$. These bounds in conjunction with *a priori* approximations of $\mathrm{E}[XZ]$ and $\mathrm{E}[Z^2]$ led to an equation expressed in terms of known system constants, bounded parameters, and the probability of being in the plastic state, $P\{Z = u\}$, which was then solved numerically to determine bounds on $P\{Z = u\}$. With bounds on $P\{Z = u\}$ established, very accurate approximations of $\sigma_{\dot{X}}$ and σ_X were evaluated as functions of $P\{Z = u\}$.

The primary usefulness of the derived expressions is that they provide a quick and easy way to determine very accurate stationary response statistics of a bilinear hysteretic oscillator for a wide array of system parameters which in turn may be used in rudimentary design or analysis considerations.

9. ACKNOWLEDGEMENT

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